SOLUTION OF NON-LINEAR EQUATIONS BY ITERATIVE METHODS

(Bisection method and False – position method)

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Introduction - An expression of the form

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n ,$$

where, a_0 , a_1, a_2, ……, a_{n-1}, a_n are constants , independent of x , and ' n' is a positive integer is called a polynomial in x of degree n provided $a_0 \neq 0$

- Any value of x which makes polynomial $P_n(x)$ equal to zero is called a zero or root of polynomial $P_n(x)$.
- Every polynomial of degree 'n' has exactly 'n' zero's or roots
- Graphically , the root (zero) of the polynomial $P_n(x)$ is the value of x where the graph of polynomial $P_n(x)$ crosses the x-axis .
- The equations may be either Algebraic or Transcendental.

Algebraic Equation - An equation in the form

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$, where a_0 , a_1 , a_2, …… , a_n are constants , $a_0 \neq 0$, & n is a positive integer is called an Algebraic equation, if the corresponding f(x) is a polynomial.

- Algebraic equations have real or complex roots.

  e.g. - $x^3 + 2x^2 - 9x + 8 = 0$ is an Algebraic equation having 3 roots , as value of n is 3 , i.e. a third degree polynomial equation has 3 values of x at which polynomial becomes zero.

Transcendental Equation - An Algebraic equation

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$, is said to be

Transcendental equation, if f(x) contains trigonometric, or logarithmic or exponential functions.

e.g. - 1) $2\sin x - x = 3$. 2) $e^x \cos x - 1.2\sin x + 4x = 0$.

There are many methods used to obtain root of an algebraic & transcendental equations. The solution of an equation $ax^2 + bx + c = 0$. $(a \neq 0)$ analytically is given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .There is also trial & error method or graphical method. However, the use of iterative methods or use of computer gives the solution (root) of an equation very easily.

Iterative method - Iterative method or indirect method is based on the concept of successive approximation. In this method we start with initial approximation to the root and then obtain sequence of iterates x_k which tends to the actual or true solution to the root, Iterative method is further classified as

Bracketing method or Interpolation method : In this method we assume two guesses in such a way That a desired root lies between these two guesses. Therefore we restrict the root or bract the root. There are varies bracketing method namely

- Bisection method
- False – position method (Regula Falsi method)

Open end method (open method) : In this method we assume single guess value which is closer to the required root . We do-not restrict the root or bracket the root. Following are the open methods.

- Secant method
- Newton - Raphson method
- Successive approximation method

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I. Bisection Method (Half – Interval method or Interval halving method or binary chopping method or Bolzano's method)

Bisection method is one of the bracketing methods & most easiest iterative method for finding the roots of an equation. This method is based on the intermediate theorem for continuous function.

Procedure : Let $f(x) = 0$ be an equation where $f(x)$ is a continuous function on $[a, b]$. Suppose we want to find the solution (root) of an equation $f(x) = 0$

We assume two initial guess a_0 & b_0 in $[a, b]$ such that $f(a_0)$ & $f(b_0)$ have opposite signs i.e. $f(a_0) . f(b_0) < 0$. Then $f(x)$ has at least one root, say x_1 in $[a_0, b_0]$

Step- 1. Compute $x_1 = \frac{a_0 + b_0}{2}$ to get first approximation root of an equation $f(x) = 0$ & then compute $f(x_1)$.

If $f(a_0).f(x_1) = 0$, then x_1 is the required root of the equation.

If $f(a_0). f(x_1) > 0$, then the root lies in the right - half interval $[x_1, b_0]$. In this case we set $a_1 = x_1$ & $b_1 = b_0$. Else if $f(a_0).f(x_1) < 0$, then the root lies in the left - half interval $[a_0, x_1]$. In this case we set $a_1 = a_0$ & $b_1 = x_1$.

Step – 2. Compute $x_2 = \frac{a_1 + b_1}{2}$ to get second approximation root of an equation $f(x) = 0$ & then compute $f(x_2)$.

If $f(a_1) \cdot f(x_2) = 0$, then x_2 is required root of an equation.

If $f(a_1) \cdot f(x_2) > 0$, then the root lies in the right - half interval $[x_2, b_1]$. In this case we set $a_2 = x_2$ & $b_2 = b_1$. Else if $f(a_1).f(x_2) < 0$, then the root lies in the left - half interval $[a_1, x_2]$. In this case we set $a_2 = a_1$ & $b_2 = x_2$.

Continue the procedure till the number of iterations or till the desired accuracy is achieved.

Example1 : Do four iterations to find the root of an equation

$$f(x) = x^3 + x - 1 = 0 \text{ using bisection method.}$$

Solution: Given $f(x) = x^3 + x - 1 = 0$.

By guess, note that, $f(0) = -1 < 0$ & $f(1) = 1 > 0$

$\therefore$ $f(0) \cdot f(1) < 0$ (negative) & hence there is at least one root say $x_1 \in [0, 1]$. We set $a_0 = 0$ & $b_0 = 1$

__**Iteration**__ **1** - For $a_0 = 0$ & $b_0 = 1$, we have $x_1 = \frac{a_0 + b_0}{2} = \frac{0+1}{2} = 0.5$.

Then, $f(x_1) = f(0.5) = -0.375$

Since, $f(a_0) \cdot f(x_1) > 0$ (i.e. positive) & hence root lies in the right - half interval $[x_1, b_0]$. We set $a_1 = x_1 = 0.5$ & $b_1 = b_0 = 1$

__**Iteration**__ **2** - For $a_1 = 0.5$ & $b_1 = 1$, we have $x_2 = \frac{a_1 + b_1}{2} = \frac{0.5+1}{2} = 0.75$.

Then, $f(x_2) = f(0.75) = 0.1718$

Since, $f(a_1) \cdot f(x_2) < 0$ (i.e. negative) & hence root lies in the left half interval $[a_1, x_2]$. We set $a_2 = a_1 = 0.5$ & $b_2 = x_2 = 0.75$.

__**Iteration**__ **3** - For $a_2 = 0.5$ & $b_2 = 0.75$, we have

$$x_3 = \frac{a_2+b_2}{2} = \frac{0.5+0.75}{2} = 0.625.$$

Then, $f(x_3) = f(0.625) = -0.1306$

Since, $f(a_2) \cdot f(x_3) > 0$ (i.e. positive) & hence root lies in the right - half interval $[x_3, b_2]$. We set $a_3 = x_3 = 0.625$ & $b_3 = b_2 = 0.75$.

<u>Iteration</u> 4 - For $a_3 = 0.625$ & $b_3 = 0.75$, we have

$$x_4 = \frac{a_3+b_3}{2} = \frac{0.625+0.75}{2} = 0.6875$$

Then, $f(x_4) = f(0.6875) = 0.01245$.

Since, $f(a_3) \cdot f(x_4) < 0$ (i.e. negative) & hence root lies in the left half interval $[a_3, x_4]$. We set $a_4 = a_3 = 0.625$ & $b_4 = x_4 = 0.6875$.

<u>Iteration</u> 5 - For $a_4 = 0.625$ & $b_4 = 0.6875$, we have

$$x_5 = \frac{a_4+b_4}{2} = \frac{0.625+0.6875}{2} = 0.6562$$

Thus, The approximate value of the required root after 5 iteration is $x_5 \cong 0.6562$

Example2 : Find the root of the equation $f(x) = xe^x - \cos x = 0$ using bisection method upto two decimal places.

Solution: Given $f(x) = xe^x - \cos x = 0$.

By guess, note that, $f(0) = -1 < 0$ & $f(1) = 2.1779 > 0$

$\therefore$ $f(0) \cdot f(1) < 0$ (negative) & hence there is at least one root say $x_1 \in [0, 1]$. We set $a_0 = 0$ & $b_0 = 1$

<u>Iteration</u> 1 - For $a_0 = 0$ & $b_0 = 1$, we have $x_1 = \frac{a_0+b_0}{2} = \frac{0+1}{2} = 0.5$.

Then, $f(x_1) = f(0.5) = -0.05322$

Since, $f(a_0) \cdot f(x_1) > 0$ (i.e. positive) & hence root lies in the right - half interval $[x_1, b_0]$. We set $a_1 = x_1 = 0.5$ & $b_1 = b_0 = 1$

<u>Iteration</u> 2 - For $a_1 = 0.5$ & $b_1 = 1$, we have $x_2 = \frac{a_1 + b_1}{2} = \frac{0.5 + 1}{2} = 0.75$.

Then, $f(x_2) = f(0.75) = 0.8560$

Since, $f(a_1) \cdot f(x_2) < 0$ (i.e. negative) & hence root lies in the left - half interval $[a_1, x_2]$. We set $a_2 = a_1 = 0.5$ & $b_2 = x_2 = 0.75$.

<u>Iteration</u> 3 - For $a_2 = 0.5$ & $b_2 = 0.75$, we have

$$x_3 = \frac{a_2 + b_2}{2} = \frac{0.5 + 0.75}{2} = 0.625.$$

Then, $f(x_3) = f(0.625) = 0.3566$

Since, $f(a_2) \cdot f(x_3) < 0$ (i.e. negative) & hence root lies in the right - half interval $[a_2, x_3]$. We set $a_3 = a_2 = 0.5$ & $b_3 = x_3 = 0.625$.

<u>Iteration</u> 4 - For $a_3 = 0.5$ & $b_3 = 0.625$, we have

$$x_4 = \frac{a_3 + b_3}{2} = 0.5625$$

Then, $f(x_4) = f(0.5625) = 0.1412$.

Since, $f(a_3) \cdot f(x_4) < 0$ (i.e. negative) & hence root lies in the left- half interval $[a_3, x_4]$. We set $a_4 = a_3 = 0.5$ & $b_4 = x_4 = 0.5625$.

<u>Iteration</u> 5 - For $a_4 = 0.5$ & $b_4 = 0.5625$, we have

$$x_5 = \frac{a_4 + b_4}{2} = 0.5312$$

Then, $f(x_5) = f(0.5312) = 0.04135$.

Since, $f(a_4) \cdot f(x_5) < 0$ (i.e. negative) & hence root lies in the left - half interval $[a_4, x_5]$. We set $a_5 = a_4 = 0.5$ & $b_5 = x_5 = 0.5312$.

Iteration 6 - For $a_5 = 0.5$ & $b_5 = 0.5312$, we have

$$x_6 = \frac{a_5 + b_5}{2} = 0.5156$$

Then, $f(x_6) = f(0.5156) = -0.00647$.

Since, $f(a_5) \cdot f(x_6) > 0$ (i.e. positive) & hence root lies in the right - half interval $[x_6, b_6]$. We set $a_6 = x_6 = 0.5156$ & $b_6 = b_5 = 0.5312$

Iteration 7 - For $a_6 = 0.5156$ & $b_6 = 0.5312$, we have

$$x_7 = \frac{a_6 + b_6}{2} = 0.5234$$

Then, $f(x_7) = f(0.5234) = 0.017$.

Since, $f(a_6) \cdot f(x_7) < 0$ (i.e. negative) & hence root lies in the right - half interval $[a_6, x_7]$. We set $a_7 = a_6 = 0.5156$ & $b_7 = x_7 = 0.5234$.

Iteration 8 - For $a_7 = 0.5156$ & $b_7 = 0.5234$, we have

$$x_8 = \frac{a_7 + b_7}{2} = 0.5195$$

Then, $f(x_8) = f(0.5195) = 0.00540$.

Since, $f(a_7) \cdot f(x_8) < 0$ (i.e. negative) & hence root lies in the right - half interval $[a_7, x_8]$. We set $a_8 = a_7 = 0.5156$ & $b_8 = x_8 = 0.5195$.

Iteration 9 - For $a_8 = 0.5156$ & $b_8 = 0.5195$, we have

$$x_9 = \frac{a_8 + b_8}{2} = 0.5175$$

Since 8^{th} and 9^{th} iterations are same upto two decimal places, so root is 0.51 approximately.

II. False Position Method (Regula-Falsi Method): False position method is Bracketing method (interpolation method). This method is similar to the Bisection. As in Bisection method, in this method also we choose two points (initial Guesses) x_0 and x_1 such that f (x_0) and f(x_1) are of opposite signs i.e. f(x_0).f (x_1) < 0 (negative) then by intermediate value theorem root lies between x_0 and x_1

Procedure : Let f (x) = 0 be an equation where f (x) is a continuous function. Suppose we want to find the solution (root) of an equation f (x) = 0

We assume two initial guess x_0 and x_1 such that f(x_0) and f(x_1) are of opposite signs i.e. f(x_0).f (x_1) < 0 (negative) then by intermediate value theorem root lies between x_0 and x_1.

<u>Step</u>1: We obtain the first iterate value of the root by using the same rule (formula) of secant method.

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Then find the value of f(x_2) for this x_2, if f(x_2) = 0 , then x_2 is the required root. If f(x_0).f(x_2) > 0, then the root lies between x_1 and x_2. In this case we set, $x_0 = x_2$ and compute the next iteration value

$$x_3 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Else, if $f(x_0).f(x_2) < 0$ then root lies between x_0 and x_2. In this case we set $x_1 = x_2$ and compute

$$x_3 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

<u>Step</u> 2: Continue the procedure till the number of iterations or till desired accuracy is achieved.

Example 1: Find a real root of the equation $f(x) = x^3 - 2x - 5$ by the method of false poition upto three places of decimals.

Solution : Given $f(x) = x^3 - 2x - 5 = 0$.

Now $f(2) = -1$ and $f(3) = 16$.

Since $f(2).f(3) < 0$ (negative) and hence root lies between 2 and 3.

Here we set $x_0 = 2$ and $x_1 = 3$

Step 1 : Now we compute

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.05882$$

$\Rightarrow f(x_2) = f(2.05882) = (2.05882)^3 - 2(2.05882) - 5 = -0.3908.$

Since $f(x_0).f(x_2) > 0$ and hence root lies between x_2 and x_1

Here ,we set $x_0 = x_2 = 2.05882$ and $f(x_0) = f(x_2) = -0.3908$.

Step 2: comput $x_3 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.0588(16) - 3(-0.3908)}{16 - (-0.3908)} = 2.0812$

$\Rightarrow$ f (x_3) = f (2.0812) = $(2.0812)^3$- 2 (2.0812) – 5 = – 0.1479.

Since f (x_0).f (x_3) > 0 , hence root lies between x_3 = 2.0812 & x_1 = 3

Here we set $x_0 = x_3 = 2.0812$

Step 3: By taking $x_0 = x_3 = 2.0812$, $x_1 = 3$, f (x_3) = – 0.1479

f $(x_1) = 16$

$$compute\ x_4 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.0812\,(16) - 3\,(-0.148)}{16 - (-0.148)} = 2.0897$$

$\Rightarrow$ f (x_4) = f(2.0897) = $(2.0897)^3$ – 2 (2.0897) – 5 = – 0.05400

Since f $(x_0).f(x_4) > 0$ *and* hence root lies between x_4 = 2.0897 and x_1 = 3. Here we set , $x_0 = x_4 = 2.0897$,

Step 4: By taking $x_0 = x_4 = 2.0897$, $x_1 = 3$, f (x_4) = – 0.05400

$$\text{compute } x_5 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.0897\,(16) - 3(-0.5400)}{16 - (-0.5400)} = 2.0926$$

$\Rightarrow$ f (x) = f (2.0926) = $(2.0926)^3$ – 2 (2.0926) – 5 = – 0.0217

Since f (x_0). f (x_5) > 0 and hence root lies between x_5 and x_1

Here we set , $x_0 = x_5 = 2.0926$

Step 5: By taking $x_0 = x_5 = 2.0926$, $x_1 = 3$, f (x_5) = 2.0926

$$\text{Compute } x_6 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.0926\,(16) - 3(-0.0217)}{16 - (-0.2170)} = 2.0938$$

$\Rightarrow$ f (x_6) = f (2.0938) = $(2.0938)^3$ - 2(2.0938) – 5 = – 0.0083

Since f (x_0) . f (x_6) > 0 and hence root lies between x_6 and x_1

Here we set $x_0 = x_6$ = 2. 0938

Step 6: By taking $x_0 = x_6 = 2.0938$, f (x_6) = - 0.0083

Compute $x_7 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.0938(16) - 3(-0.0083)}{16-(-0.0083)} = 2.0942$

$\Rightarrow f(x_7) = (2.0942) = (2.0942)^3 - 2(2.0942) - 5 = -0.0039$

Since $f(x_0).f(x_7) > 0$ and hence root lies between x_7 and x_1

Here we set , $x_7 = 2.0942$ and $x_1 = 3$

Step 7: By taking $x_0 = x_7 = 2.0942$, $x_1 = 3$, $f(x_7) = 2.0942$

Compute $x_8 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.0942(16) - 3(-0.0039)}{16-(-0.0039)} = 2.0944$

Hence , the required root correct to three decimal places is 2.094 approximately .

Example 2: Find a real root of the equation $xe^x = 2$ by the method of false which lies in the range{0,1}.

Solution : Given $f(x) = xe^x - 2 = 0$.

Now f (0.8) = – 0.2196, f (0.9) = 0.2136, f (0.852) = – 00263.

and f (0.853) = 001715.

Hence root lies between 0.852 and 0.853.

Here we set $x_0 = 0.852$ and $x_1 = 0.853$

Step 1 : Now we compute

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 0.8526052$$

$\Rightarrow f(x_2) = f(2.8526052) = -0.0000009$.

Since $f(x_0).f(x_2) > 0$ and hence root lies between x_1 and x_2

Here ,we set $x_0 = x_2 = 2.8526052$ and $f(x_0) = f(x_2) = -0.0000009$.

Step 2: comput $x_3 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$ = 0.8526055

Since x_2 and x_3 are same upto four decimal places, so the root correct to four decimal places is 0.8526.

References-

1. An Introduction to Numerical Analysis – Devi Prasad.

2. Introductory Methods of Numerical Analysis – S.S.Sastry.