SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS BY NEWTON – RAPHSON METHOD AND SECANT METHOD

DR. H. T. DINDE

DEPARTMENT OF MATHEMATICS

KARMAVEER BHAURAO PATIL COLLEGE, URUN- ISLAMPUR

htdmaths@gmail.com

1. Newton - Raphson Method: The Newton - Raphson method is the type of open method (Extrapolation method). It is powerful technique for solving algebraic and transcendental equation $f(x) = 0$, numerically.

Procedure: – Let x_0 be an initial approximate value of the root of the equation $f(x) = 0$ and let h be sufficiently small real number. If $\xi = x_0 + h$ be exactly value of the root of the equation $f(x) = 0$ then, $f(\xi) = 0$ or $f(x_0 + h) = 0$.

Now, expanding $f(x_0 + h)$ by Taylor's series , we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots\ldots\ldots = 0.$$

Since, h is very small and hence neglecting the terms containing h^2 and higher power terms of h, we get

$$\therefore \quad f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow \quad h = \frac{-f(x_0)}{f'(x_0)}$$

Thus, the first approximation to the root is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, taking x_1 as initial approximation, a still better approximation x_2 is given by,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Continuing, the procedure n times, we get better approximation to the root , which is given by ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} , \qquad n = 0,1, 2, 3, \ . \ . \ .$$

This is known as Newton – Raphson formula or Newton's Iteration formula.

Example1: Find the real root of the equation $f(x) = x^4 - x - 10 = 0$ up to three decimal places by using Newton – Raphson method.

Solution: Given $f(x) = x^4 - x - 10 = 0$ & $f'(x) = 4x^3 - 1$

$$\therefore \quad f(1) = -10 \quad \& \quad f(2) = 4.$$

Hence, root lies between 1 and 2.

Let us assume $x_0 = 1.6$

Then by Newton Raphson Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} , \qquad n = 0, 1, 2, 3, \ . \ . \ .$$

$$x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} = \frac{3x_n^4 + 10}{4x_n^3 - 1} , \qquad n = 0, 1, 2, 3, \ . \ . \ .$$

<u>Iteration</u> 1 - For $n = 0$, $x_0 = 1.6$

$$x_1 = \frac{3(x_0)^4 + 10}{4(x_0)^3 - 1} = \frac{3(6.5536) + 10}{4(4.096) - 1} = \frac{29.66}{15.38} = 1.9284$$

<u>Iteration</u> 2 - For $n = 1$, $x_1 = 1.9284$

$$x_2 = \frac{3(x_1)^4 + 10}{4(x_1)^3 - 1} = \frac{3(13.8289)+10}{4(7.1711)-1} = \frac{51.4867}{27.6847} = 1.8597$$

$$x_2 = 1.8597$$

Iteration 3 - For $n = 2$, $x_2 = 1.8597$

$$x_3 = \frac{3(x_2)^4 + 10}{4(x_2)^3 - 1} = \frac{3(11.9611)+10}{4(6.4317)-1} = \frac{45.8833}{24.7268} = 1.8556$$

Iteration 4 - For $n = 3$ $x_3 = 1.8556$

$$x_4 = 1.8555$$

Hence, the required root of given equation $x^4 - x - 10 = 0$ up to three decimal places is 1.855.

Example2: Find the real root of the equation $3x - \cos x - 1 = 0$ by Newton – Raphson method.

Solution: Given $f(x) = 3x - \cos x - 1 = 0$ & $f'(x) = 3 + \sin x$

Now, $f(0) = -2$ & $f(1) = 3 - \cos 1 - 1 = 3 - 0.5403 - 1 = 1.4597$.

Hence, root lies between 0 and 1.

Let us assume $x_0 = 0.6$

Then by Newton Raphson Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3, \ldots$$

$$x_{n+1} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}, \qquad n = 0, 1, 2, 3, \ldots$$

Iteration 1 - For $n = 0$, $x_0 = 0.6$

$x_1 = 0.6071$

Iteration 2 - For $n = 1$, $x_1 = 0.6071$

$x_2 = 0.6071$

Here, $x_1 = x_2$, so the root correct to four decimal places is 0.6071.

2. Secant Method : Secant method is an open method (Extrapolation method). In this method we choose or use two initial guesses (or two numbers) x_0 *and* x_1 close to the root 'ξ' (or x) of the equation f(x) = 0 such that $f(x_0) \neq f(x_1)$ [In this method $f(x_0)$ & $f(x_1)$ are not necessarily of opposite signs].

As a next approximation x_2 is obtained as the point intersection of y = 0 (i.e. x-axis) and a chord passing through the points $(x_1, f(x_1))$.

By putting y = 0 in the equation of chord

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$$

We get,

$$x_2 = x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general the iterative sequence $\{x_n\}$ of roots of the equation f(x) = 0 is computed by the rule.

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad n = 1, 2, 3, \ldots$$

This sequence converges to the root ξ (or x) of the equation f (x) = 0 i.e. $f(\xi) = 0$.

Example1: Do three iteration of secant method to find the root of equation $f(x) = x^3 - 3x + 1$, taking $x_0 = 1$, $x_1 = 0.5$.

Solution : Given $f(x) = x^3 - 3x + 1 = 0$ & $x_0 = 1$ $x_1 = 0.5$.

To find root of given equation by secant method we use the following formula

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad ....(1)$$

Iteration I : For n = 1 in (1) , we have

$f(x_0) = f(1) = -1$ & $f(x_1) = f(0.5) = -0.375$

$$\therefore x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1.(-0.375) - 0.5(-1)}{-0.375 - (-1)} = 0.2$$

Iteration II : For n = 2 in (1) we have

$f(x_1) = f(0.5) = -0.375$ & $f(x_2) = f(0.2) = 0.408$

$$\therefore x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.5(0.408) - 0.2(-0.375)}{0.408 - (-0.375)} = 0.3563$$

Iteration III : For n = 3 in (1) we have

$f(x_2) = f(0.2) = 0.408$ & $f(x_3) = -0.02366$

$$\therefore x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{0.2(-0.02366) - 0.3563(0.408)}{-0.02366 - 0.408} = 0.3477$$

Thus , The approximate value of the required root after 3 iterations is $x_3 \cong 0.3477$.

Example2: Find a root of the equation $x - e^{-x} = 0$ correct to three decimal places by the secant method.

Solution : Given $f(x) = x - e^{-x} = 0$.

Then, $f(0) = -1$ & $f(1) = 1 - e^{-1} = 0.6321$

Taking $x_0 = 0$, $x_1 = 1$.

To find root of given equation by secant method we use the following formula

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad \ldots\ldots.(1)$$

Iteration I : For n = 1 in (1) , we have

$f(x_0) = f(0) = -1$ & $f(x_1) = f(1) = 0.6321$

$$\therefore x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1}{0.6321 + 1} = \frac{1}{1.6321} = 0.6127$$

Iteration II : For n = 2 in (1) we have

$f(x_1) = f(1) = 0.6321$ & $f(x_2) = f(0.6127) = 0.0708$

$\therefore x_3 = 0.5639$

Iteration III : For n = 3 in (1) we have

$f(x_2) = f(0.6127) = 0.0708$ & $f(x_3) = f(0.5639) = -0.0051$

$\therefore x_4 = 0.5672$

Iteration IV : For n = 4 in (1) we have

$f(x_3) = f(0.5639) = -0.0051$ & $f(x_4) = f(0.5672) = 0.0001$

$\therefore x_4 = 0.5670$

Thus , The approximate value of the required root correct to three decimal places is 0.5670.

References-

1. An Introduction to Numerical Analysis – Devi Prasad.

2. Introductory Methods of Numerical Analysis – S.S.Sastry.